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Induced Hopf stucture and irreducible representations of an elliptic $\mathcal{U}_{q,p}(sl(2))$ via a nonlinear map

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Abstract

Shiraishi's two parameter generalization of $\mathcal{U}_q(sl(2))$ to $\mathcal{U}_{q,p}(sl(2))$ involving an elliptic function is considered. The generators are mapped non-linearly on those of $\mathcal{U}_q(sl(2))$. This gives directly the irreducible representations and an induced Hopf structure. This is one particular example of the scope of a class of non-linear maps introduced by us recently.

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Recently Shiraishi [1] has made what he describes as " an attempt at obtaining an elliptic $sl(2)$ algebra...". He generalizes the $\mathcal{U}_q(sl(2))$ algebra as follows

$$q^{2J_0}\hat{J}_\pm q^{-2J_0} = q^{\pm 2}\hat{J}_\pm \tag{1}$$

$$[\hat{J}_+, \hat{J}_-] = \sum_{n\in Z}(-1)^n q^{2J_0(2n+1)}p^{(n+1/2)^2} \tag{2}$$

where for his (t, e, f) we have written $(q^{2J_0}, \hat{J}_+, \hat{J}_-)$ respectively. The generalization involves a theta function on the rhs of (2) insead of $[2J_0]$ (i.e. $(q^{2J_0} - q^{-2J0})(q - q^{-1})^{-1}$) giving standard $\mathcal{U}_q(sl(2))$. This is proposed in the context of the "elliptic algebra" $\mathcal{A}_{q,p}(\hat{s}l(2))$ of [2]. The connection between the two formalismes ([1] and [2]) is not clear. But the algebra given by (1) and (2) (which will be denoted by $\mathcal{U}_{p,q}(sl(2))$ has some interesting properties. A Heisenberg-Clifford realization is presented in [1].

We point out in this note that non-linear mappings of [3] can be adapted to provide one between $\mathcal{U}_q(sl(2)$ and $\mathcal{U}_{p,q}(sl(2))$. Such a map immediately gives the irreducible representations of $\mathcal{U}_{p,q}(sl(2))$ and provides an induced Hopf stucture(absent in [1]).

Let us recapitulate the formalisme of Sec.3 of [3] in a form well-adapted to the present case. The generators of $\mathcal{U}_q(sl(2))$ satisfy

$$q^{2J_0}J_\pm q^{-2J_0} = q^{\pm 2}J_\pm \tag{3}$$

$$[J_+, J_-] = [2J_0] \tag{4}$$

The Casimir operator is

$$C = J_-J_+ + [J_0][J_0+1] \tag{5}$$

Define (introducing a function ϕ with suitable properties to be specified below),

$$\hat{J}_+ = J_+\Big(\frac{\phi(C) - \phi([J_0][J_0+1])}{C - [J_0][J_0+1]}\Big)^{\frac{1+\eta}{2}} \tag{6}$$

$$\hat{J}_- = \Big(\frac{\phi(C) - \phi([J_0][J_0+1])}{C - [J_0][J_0+1]}\Big)^{\frac{1-\eta}{2}}J_- \tag{7}$$

where $\eta = 0, 1, -1$. (The choice $\eta = 0$ will be the standard one. But in certain contexts it might be prefable to avoid squareroots by setting $\eta = 1$ or -1. In the latter cases the familiar matrix elements of$J_\pm$ should also be consistently written as $([j][j+1] - [m][m\pm 1])^{\frac{1\pm\eta}{2}}$.)

It follows easily (for any η and using (5)) that

$$[\hat{J}_+, \hat{J}_-] = \phi([J_0][J_0+1]) - \phi([J_0][J_0-1]) \tag{8}$$

Hence for any suitably chosen function $\chi(J_0)$ to ensure

$$[\hat{J}_+, \hat{J}_-] = \chi(J_0) \tag{9}$$

one must have

$$\phi([J_0][J_0+1]) - \phi([J_0][J_0-1] = \chi(J_0) \tag{10}$$

This is a general result. In [3] we gave the example that for

$$\chi(J_0) = [2J_0](1+\beta[2][J_0]^2) \tag{11}$$
$$\phi(x) = x + \beta x^2 \tag{12}$$

For the general case, it is convenient to define

$$\phi([J_0][J_0+1]) = \psi(J_0) = \sum_{n\in Z} a_n q^{2nJ_0} \tag{13}$$

limiting our considerations to such a series in $t^{\pm 1} = q^{\pm 2J_0}$. It is not essential to reconvert $\psi(J_0)$ into $\phi([J_0][J_0+1])$.

Similarly one may define (in (6) and (7))

$$\phi(C) = \psi(J_{op}) \tag{14}$$

where

$$C = [J_{op}][J_{op}+1] \tag{15}$$

giving $q^{2J_{op}}$ as the solution of a quadratic equation.

Let in (9) and (10)

$$\chi(J_0) = \sum_{k\geq 1}(b_k q^{2kJ_0} + b_{-k} q^{-2kJ_0}) \tag{16}$$

From (10), (13) and (16)

$$a_k = \frac{q^k b_k}{(q^k - q^{-k})}, \qquad a_{-k} = -\frac{q^{-k} b_{-k}}{(q^k - q^{-k})} \tag{17}$$
$$(k = 1, 2, \cdots) \tag{18}$$

while a_0 in (13) is arbitrary (and indeed cancels out in the difference $(\psi(J_{op}) - \psi(J_0))$. Hence

$$\psi(J_0) = a_0 + \sum_{k\geq 1} \frac{(b_k q^{k(2J_0+1)} - b_{-k} q^{-k(2J_0+1)})}{(q^k - q^{-k})} \tag{19}$$

Choosing, for example, c_0 being a finite constant,

$$a_0 = c_0 - \sum_{k\geq 1} \frac{(b_k - b_{-k})}{(q^k - q^{-k})} \tag{20}$$

$\psi(J_0)$ can be, separately, given a finite limit as $q \to 1$. (But this is not strictly necessary as a_0 cancels in the numerator.)

Writing (12) in the form (13) (11 in the form(16)) one has

$$a_{\pm 1} = \frac{q^{\pm 1}}{(q-q^{-1})^2}(1 - \frac{2\beta}{(q-q^{-1})^2}) = \pm \frac{q^{\pm 1}}{(q-q^{-1})} b_{\pm 1} \tag{21}$$

$$a_{\pm 2} = \frac{q^{\pm 2}}{(q-q^{-1})^4} \frac{\beta}{(q+q^{-1})} = \pm \frac{q^{\pm 2}}{(q^2-q^{-2})} b_{\pm 2} \tag{22}$$

Corresponding to (2) one has

$$\begin{aligned} \chi(J_0) &= \sum_{n \in Z} (-1)^n q^{2j_0(2n+1)} p^{(n+\frac{1}{2})^2} \\ &= \sum_{k \geq 1} (b_k q^{2kJ_0} + b_{-k} q^{-2kJ_0}) \end{aligned} \tag{23}$$

with for even and odd k respectively

$$b_{\pm k} = 0, \qquad b_{\pm k} = (-1)^{\frac{k \pm 1}{2}} p^{(\frac{k}{2})^2}. \tag{24}$$

Now (17) gives the corresponding $a_{\pm k}$'s.

Considering generic q, our map gives, quite generally, the $(2j+1)$ dimentional irreducible representations (for (half) integer j)

$$q^{\pm 2J_0} \mid j, m\rangle = q^{\pm 2m} \mid j, m\rangle \qquad (m = j, j-1, \cdots, -j) \tag{25}$$

$$\hat{J}_{\pm} \mid j, m\rangle = (\psi(j) - \psi(m))^{\frac{1 \pm \eta}{2}} \mid j, m \pm 1\rangle \tag{26}$$

$$= \Big(\phi([j][j+1]) - \phi([m][m+1]\Big)^{\frac{1 \pm \eta}{2}} \mid j, m \pm 1\rangle \tag{27}$$

$$(\eta = 0, 1, -1) \tag{28}$$

It is convenient to express the Casimir now as

$$\hat{C} = \hat{J}_- \hat{J}_+ + \phi([J_0][J_0+1]) \tag{29}$$

so that

$$\hat{C} \mid j, m\rangle = \phi([j][j+1]) \mid j, m\rangle \tag{30}$$

$$= \psi(j) \mid j, m\rangle \tag{31}$$

For $\mathcal{U}_q(sl(2))$ one has the coproducts

$$\Delta J_0 = J_0 \otimes 1 + 1 \otimes J_0 \tag{32}$$

$$\Delta J_{\pm} = J_{\pm} \otimes q^{J_0} + q^{-J_0} \otimes J_{\pm} \tag{33}$$

$$\Delta C = (\Delta J_-)(\Delta J_+) + [\Delta J_0][\Delta J_0 + 1] \tag{34}$$

They induce the coproducts

$$\Delta\hat{J}_+ = \Delta J_+ \Big(\frac{\phi(\Delta C) - \phi([\Delta J_0][\Delta J_0 + 1])}{\Delta C - [\Delta J_0][\Delta J_0 + 1]} \Big)^{\frac{1+\eta}{2}} \tag{35}$$

$$\Delta\hat{J}_- = \Big(\frac{\phi(\Delta C) - \phi([\Delta J_0][\Delta J_0 + 1])}{(\Delta C) - [\Delta J_0][\Delta J_0 + 1]} \Big)^{\frac{1-\eta}{2}} \Delta J_- \tag{36}$$

The counits and the antipodes are analogously induced. The results from (25) to (36) are general. For each particular case one has to implement the appropriate ψ or ϕ as discussed above.

In (25) to (27) we have considered $(2j + 1)$ dimentional irreps for generic q. But our formalism can be implemented also for q a root of unity. The parameters j and m in (26) can have "fractional parts" [4] and one can have, for example, periodic and semiperiodic representations of entirely different dimentions. But we will not study these aspects here. The infinite series form in (2) is not appropriate for spaces of periodic representations. Here our purpose has been to draw attention to the fact that irreps and an induced Hopf structure corresponding to the type given by (2) are quite simply furnished by our nonlinear map.

A complementary class of nonlinear map has been introduced [5] relating $\mathcal{U}(sl(2))$ and $\mathcal{U}_h(sl(2))$. It is complementary in the sense that here (for $\mathcal{U}(sl(2)$ or $\mathcal{U}_q(sl(2))$ as starting point) the nonlinearity is implemented via the diagonalizable generator J_0 (or $q^{\pm J_0}$) whereas in [5] the corresponding role is played by the nondiagonalizable generator J_+. These two types of mappings can be combined. Such a study will be presented elsewhere.

Citations

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